# ABSTRACT

Two probabilistic models for 2-out-of-3 redundant system of identical units are developed. The system is considered in up-state if 2-out-of-3 units are operative. Repair facility is provided immediately whenever needed in model 1 while server takes some time to arrive at the system in model 2. The distribution of repair and waiting time are taken as general with different probability density functions (pdf) whereas failure time distribution of the unit is negative exponential. Regenerative point technique is adopted to derive the expressions for some reliability and economic measures. Graphs are plotted to discuss reliability and cost-benefit analysis of the models for a particular case. The applications of the present models can be seen in computer, electrical and mechanical systems. The paper suggests that 2-out-of-3 redundant systems can be made more reliable and profitable by providing components of low failure rates vand immediate repair facility.

Reliability And Cost-benefit Analysis

of 2-out-of-3 Redundant System
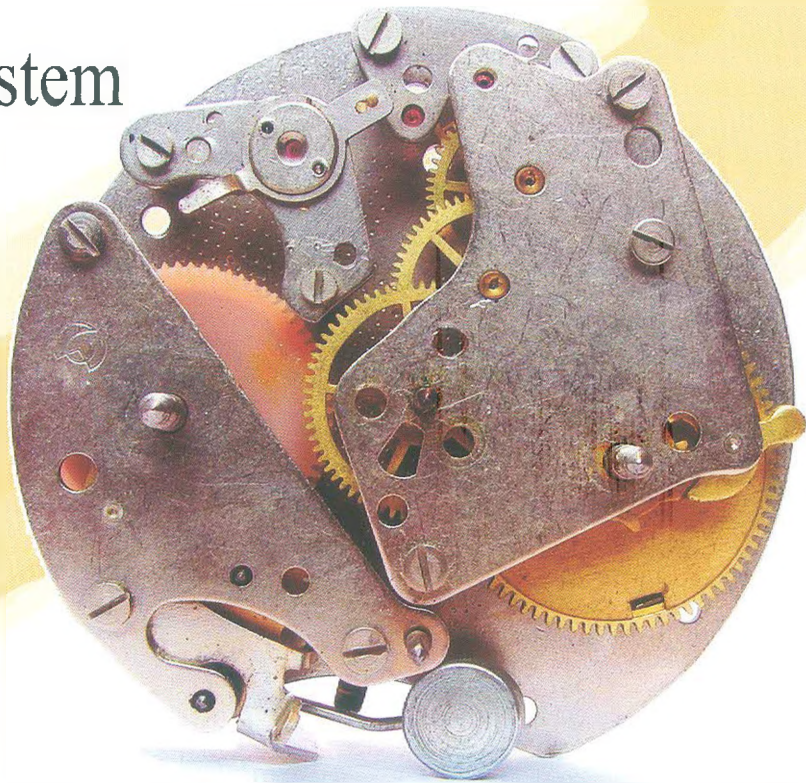
with
General Distribution of

*R*epair and *W*aiting Time

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INTRODUCTION

In the field of reliability several probabilistic models of two-unit standby systems have been discussed by obtaining reliability and/or economic measures. Chiang and Niu [1981] have evaluated the reliability of k-out-of-n : F systems. Dhillon [1992] has discussed stochastic models of k-out-of-n units family of system. Partial redundancy has also been used in many mechanical, electrical and electronic systems which function successfully if and only if at least k, 1<k<m, out of the m units are operative.

However, there may exist systems of three units in which two units work simultaneously. Ash handling plant of a fertilizer industry can be cited as a good example of such systems. In ash handling plant there are three horizontal multistage centrifugal type ash water pumps. Out of three, two pumps work simultaneously with one as cold standby.

In view of the practical applications, in industry, of three-unit redundant systems, here we study two probabilistic models for 2-out-of-3 unit redundant system of identical units. Each unit has two modes of failure-normal (N) and complete failure (F). In each model two units work in parallel and third unit is cold standby. The system is considered in up-state only if 2-out-of-3 units are operative. In Model 1, server attends the system immediately whenever needed whereas in model II, he takes sometime to arrive at the system. The server during repair cannot leave the system and unit after repair works as new. It is assumed that switches are perfect and unit in normal mode, when not working, cannot fail.

The failure time, repair time and waiting time of the server are independent and uncorrelated random variables. The failure time distribution of the unit follows negative exponential while the distributions of repair and waiting time are taken as general. Several measures of system effectiveness such as mean time to system failure (MTSF) steady state availability, busy period and expected number of units by the server are obtained by using semi-markov processes and regenerative-point technique. Profit incurred to each model of the system is also obtained. Graphs are plotted to discuss reliability and cost-benefit analysis of the models for a particular case.
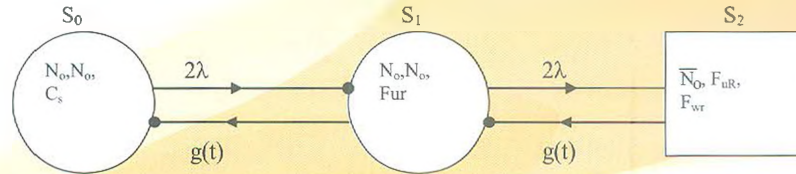
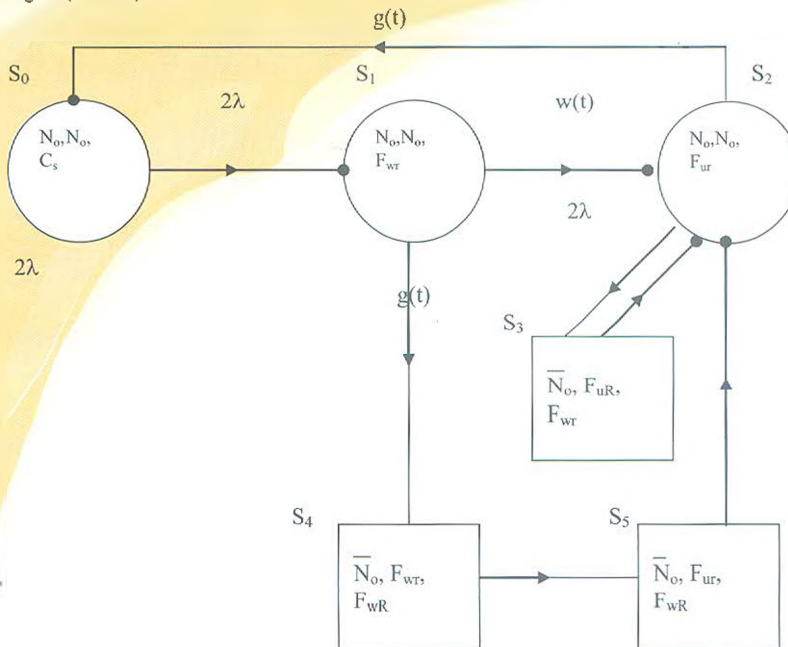## STATE-TRANSITION DIAGRAMS

Fig. 1: (Model I)

- Regenerative point
○ Operative/up-state
□ Failed/down-state

Fig. 2: (Model II)

NOTATION

E	Set of regenerative states
N_o	Unit in normal mode and operative
$\overline{N}_0$	Unit in normal mode and not operative
C_s	Unit in normal mode and cold standby
λ	Constant failure rate of an operative unit
$M_i(t)$	Probability that the system is up initially in state $S_i \in E$ is up at time t without visiting to any other regenerative state
$W_i(t)$	Probability that the server is busy in state S_i upto time t without making any transition to any other regenerative state or returning to the same via one or more non-regenerative states.
$q_{ij,kr}(t)$, $Q_{ij,kr}(t)$	p.d.f., c.d.f. of first passage time from regenerative state i to regenerative state j or to a failed state j visiting state k,r once in [0,t].
$w(t)$, $W(t)$	p.d.f., c.d.f. of waiting time of the server to arrive at the system.
$g(t)$, $G(t)$	p.d.f., c.d.f. of the repair time of the server.
$\phi_i(t)$	c.d.f. of first passage time from regenerative state i to a failed state j
$A_i(t)$	Probability that the system is in up-state at instant t given that the system entered regenerative state i at t=0.
$B_i(t)$	Probability that the server is busy at an instant time t given that the system entered the regenerative state i at t=0.
$N_i(t)$	Expected number of visits by the server in (0,t], given that the system entered the regenerative state i at t=0
F_{ur}/F_{UR}	Unit is completely failed and under repair/under repair continuously from previous state
F_{wr}/F_{wR}	Unit is completely failed and waiting for repair/waiting for repair continuously from previous state
$p_{ij}/p_{ij,kr}$	Probability of transition from regenerative state i to regenerative state j without visiting any other state in (0,t]/visiting state k,r once in (0,t], i.e. $$p_{ij} = \lim_{s \to 0} q_{ij}^*(s) \text{ and } p_{ij,kr} = \lim_{s \to 0} q_{ij}^*(s)$$
ⓢ /©	Symbol for sfieltjes convolution/Laplace convolution
LST/LT	Laplace-Stieltjes transform/Laplace transform
~ / *	Symbol for LST/LT
'	Symbol for derivative of the function

The possible transition states of the system model I and model II are shown in the following table.

State	S_0	S_1	S_2	S_3	S_4	S_5
Model I	N_o, N_o, C_s	N_o, N_o, F_{ur}	$\overline{N}_0, F_{wr}, F_{uR}$	---	---	---
Model II	N_o, N_o, C_s	N_o, N_o, F_{wr}	N_o, N_o, F_{ur}	N_o, F_{uR}, F_{wr}	$\overline{N}_0, F_{wr}, F_{WR}$	$\overline{N}_0, F_{ur}, F_{WR}$

For model I: $E=\{S_0, S_1\}$ and For model II: $E=\{S_0, S_1, S_2\}$

The possible transition states along with transition rates for model I and model II are shown in Fig. I and Fig. II respectively.

TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

Simple probabilistic considerations yield the following expressions for the non-zero elements

$$p_{ij} = Q_{ij}(\infty) = \int q_{ij}(t)dt \qquad (1)$$

For model I

$$p_{01}=1, \ p_{10}=g^*(2\lambda), \ p_{12}=1-g^*(2\lambda), \ p_{11.2}=1-g^*(2\lambda) \qquad (2)$$

clearly, $p_{01}=p_{10}+p_{12}=p_{10}+p_{11.2}=1 \qquad (3)$

For model II

$$p_{01}=1, \ p_{12}=w^*(2\lambda), \ p_{14}=1-w^*(2\lambda)=p_{12.45}, \ p_{20}=g^*(2\lambda), \ p_{23}=1-g^*(2\lambda)=p_{22.3} \qquad (4)$$

clearly, $p_{01}=1=p_{12}+p_{14}=p_{12}+p_{12.45}=p_{20}+p_{23}=p_{20}+p_{22.3} \qquad (5)$

The unconditional mean time taken by the system to transit to any regenerative state S_i when it (time) is counted from epoch of entrance into that state, is given by

$$m_{ij} = \int_0^\infty td\{Q_{ij}(t)\} = -q*'_{ij}(0)$$ (6)

and the mean sojourn time in the state S_i is given by

$$\mu_i = E(T) = \int_0^\infty P(T > t)dt$$ (7)

where T denotes the time to system failure.
Using these we have following expressions:

For model I

$$\mu_0 = m_{01}, \ \mu_1 = m_{10} + m_{12}, \ \mu'_1 = m_{10} + m_{11.2}$$ (8)

For model II

$$\mu_0 = m_{01}, \ \mu_1 = m_{12} + m_{14}, \ u_2 = m_{20} + m_{23}, \ \mu'_1 = m_{12} + m_{12.45}, \ \mu'_2 = m_{20} + m_{22.3}$$ (9)

ANALYSIS FOR MODEL I

(i) Mean time to system failure (MTSF)
On the basis of arguments used for regenerative processes, we obtain the following recursive relations for $\phi_i(t)$

$$\phi_0(t) = Q_{01}(t) \quad \phi_1(t)$$ (10)
$$\phi_1(t) = Q_{10}(t) \quad \phi_0(t) + Q_{12}(t)$$

Taking LST of above relations (10) and solving for $\tilde{\phi}_0(t)$, we get MTSF as

$$\text{MTSF (T)} = \lim_{s \to 0} \frac{1 - \tilde{\phi}_0(s)}{s} = \frac{N_{11}}{D_{11}},$$ (11)

Where, $N_{11} = \dfrac{1}{2\lambda} \{2 - g^*(2\lambda)\}$ and $D_{11} = 1 - g^*(2\lambda)$

(ii) Steady state availability
The recursive relations for $A_i(t)$ are given as
$$A_0(t) = M_0(t) + q_{01}(t) © A_1(t)$$
$$A_1(t) = M_1(t) + q_{10}(t) © A_0(t) + q_{11.2}(t) © A_1(t)$$ (12)
Where, $M_0(t) = e^{-2\lambda t}$, $M_1(t) = e^{-2\lambda t} G(t)$
Taking LT of above relations (12) and solving for $A_0^*(s)$, we get steady state availability of the system as

$$A_{10} = \lim_{s \to 0} s.A_0^*(\infty) = \frac{N_{12}}{D_{12}}$$ (13)

where, $N_{12} = \dfrac{1}{2\lambda}$ and $D_{12} = \dfrac{1}{2\lambda} - \{1 - g^*(2\lambda)\}.g^{*'}(0)$

(iii) Busy period of the server
The recursive relations for $B_i(t)$ are given as
$$B_0(t) = q_{01}(t) © B_1(t)$$
$$B_1(t) = W_1(t) + q_{10}(t) © Bo(t) + q_{11.2}(t) © B_1(t)$$ (14)
Where, $W_1(t) = \{e^{-2\lambda t} + (2\lambda e^{-2\lambda t} © 1)\}G(t)$

Taking LT of relations (14) and solving for $B_0^*(s)$, we get in long run the time for which the server is busy

$$B_{10} = \lim_{s \to 0} sB_0^*(s) = \frac{N_{13}}{D_{12}}$$ (15)

where, $N_{13} = -g^{*'}(0)$ and D_{12} is already specified

(iv)   Expected No. of visits by the server
The recursive relations for $N_i(t)$ are given as
$N_0(t) = Q_{01}(t)$ $[1+N_1(t)]$ ⓢ
$N_1(t) = Q_{10}(t)$ $N_0(t)$ ⓢ$_{1.2}(t)$ $N_1(t)$ ⓢ   (16)

Taking LST of above relations (16) and solving for $\tilde{N}_0(s)$, we get the expected number of visits per unit

time as $N_{10} = \underset{s \to 0}{lt}\ s.N_0^*\ (s) = \dfrac{N_{14}}{D_{12}}$ (17)

Where $N_{14} = g^*(2\lambda)$ and D_{12} is already specified.

ANALYSIS FOR MODEL II

(i)   Mean time to system failure (MTSF):
On the basis of arguments used for regenerative processes, we obtain following recursive relations for $\phi_i(t)$:
$\phi_0(t) = Q_{01}(t)$ $\phi_1(t)$ ⓢ
$\phi_1(t) = Q_{12}(t)$ $\phi_2(t) \times$ⓢ$Q_{14}(t)$
$\phi_2(t) = Q_{20}(t)$ $\phi_0(t) \times$ⓢ$Q_{23}(t)$ (18)

Taking LST of above relations (18) and solving for $\tilde{\phi}_0(s)$ we get MTSF as

$$MTSF(T_{20}) = \underset{s \to 0}{lt}\ \frac{1 - \tilde{\phi}_0(s)}{s} = \frac{N_{21}}{D_{21}}$$ (19)

where, $N_{21} = \dfrac{1}{2\lambda}\ [2 - g^*(2\lambda).w^*(2\lambda)]$ and $D_{21} = 1 - g^*(2\lambda).w^*(2\lambda)$

(ii)   Steady state availability
The recursive relations for $A_i(t)$ are given as
$A_0(t) = M_0(t) + q_{01}(t)$ Ⓒ $A_1(t)$
$A_1(t) = M_1(t) + \{q_{12}(t) + q_{12.45}(t)\}$Ⓒ $A_2(t)$
$A_2(t) = M_2(t) + q_{20}(t)$ Ⓒ $A_0(t) + q_{22.3}(t)$ Ⓒ $A_2(t)$ (20)
Where, $M_0(t) = e^{-2\lambda t}$, $M_1(t) = e^{-2\lambda t}\ W(t)$ and $M_2(t) = e^{-2\lambda t}\ G(t)$
Taking LT of above relations (20) and solving for $A_0^*(s)$, we get steady state availability of the system as :

                                                                                                    (21)

$A_{20} = \underset{s \to 0}{lt}\ s.A_0^*\ (s) = \dfrac{N_{22}}{D_{22}}$ where $N_{22} = \dfrac{1}{2\lambda}\ [1 + \{1 - w^*(2\lambda)\}g^*(2\lambda)]$ and

$D_{22} = \dfrac{1}{2\lambda}\ [1 + \{1 - w^*(2\lambda)\}\ \{2 - 2\lambda(g^{*'}(0) + w^{*'}(0))\}.g^*(2\lambda)] + \{1 - g^*(2\lambda)\}$
        $\{1 - 2\lambda g^{*'}(0)\}$

(iii)   Busy period of the server
The recursive relations for $B_i(t)$ are given as
$B_0(t) = q_{01}(t)$ Ⓒ $B_1(t)$
$B_1(t) = \{q_{12}(t) + q_{12.45}(t)\}$ Ⓒ $B_2(t)$
$B_2(t) = W_2(t) + q_{20}(t)$ Ⓒ $B_0(t) + q_{22.3}(t)B_2(t)$ (22)
Where, $W_2(t) = \{e^{-2\lambda t}\ (2\lambda e^{-2\lambda t}$ Ⓒ $1)\}\ G(t)$
Taking LT of relations (22) and solving for $B_0^*(s)$, we get in the long run the time for which the system is under repair as:

$B_{20} = \underset{s \to 0}{lt}\ s.B_0^*\ (s) = \dfrac{N_{23}}{D_{22}}$ (23)

where, $N_{23} = -g^{*'}(0)$ and D_{22} is already specified.

(iv)   Expected number of visits by the server
The recursive relations for $N_i(t)$ are given as

(iv)   Expected number of visits by the server

The recursive relations for $N_i(t)$ are given as

$N_0(t) = Q_{01}(t) \qquad N_1(t)$ Ⓢ

$N_1(t) = Q_{12}(t) \qquad [1+N_2(t)]$Ⓢ$+ Q_{12.45}(t) \qquad [1+N_5(t)]$Ⓢ

$N_2(t) = Q_{20}(t) \qquad N_0(t) + Q_{22.3}(t)$Ⓢ $N_2(t)$Ⓢ   (24)

Taking LST of above relations (24) and solving for $\tilde{N}_0(s)$, we get the expected number of visits per unit time as:

$$N_{20} = \lim_{s \to 0} s.\tilde{N}_0(s) = \frac{N_{24}}{D_{22}}$$ (25)

where, $N_{24} = w^*(2\lambda)$ and D_{22} is already specified.

COST-BENEFIT ANALYSIS

Profit incurred to the system models in steady state are given by

$P_{10} = K_1.A_{10} - K_2.B_{10} - K_3.N_{10} \qquad$ (For model I)

$P_{20} = K_1.A_{20} - K_2.B_{20} - K_3.N_{20} \qquad$ (For model II)   (26)

Where

K_1 = Revenue per unit up time of the system.

K_2 = Cost per unit for which server is busy.

K_3 = Cost per unit visit by the server.

PARTICULAR CASE

Let us take $g(t) = \theta.e^{-\theta t}$, $w(t) = \beta e^{-\beta t}$

We can obtain the following results:

For model I

$$MTSF(T_{10}) = \frac{N_{11}}{D_{11}}, \; Availability \; (A_{10}) = \frac{N_{12}}{D_{12}}$$

$$Busy \; period \; B_{10} = \frac{N_{13}}{D_{12}}, \; Expected \; number \; of \; visits \; (N_{10}) = \frac{N_{14}}{D_{12}}$$ (27)

For model II

$$MTSF(T_{20}) = \frac{N_{21}}{D_{21}}, \; Availability \; (A_{20}) = \frac{N_{22}}{D_{22}}$$

$$Busy \; period \; B_{20} = \frac{N_{23}}{D_{22}}, \; Expected \; number \; of \; visits \; (N_{20}) = \frac{N_{24}}{D_{22}}$$ (28)

Where,

$$N_{11} = (\theta + 4\lambda), \; N_{12} = \frac{1}{2\lambda}, \; N_{13} = \frac{1}{\theta}, \; N_{14} = \frac{\theta}{(\theta + 2\lambda)}, \; N_{21} = \frac{R_2}{S_2}, \; N_{22} = \frac{R_4}{S_3}, \; N_{23} = \frac{1}{\theta}, \; N_{24} = \frac{\theta}{(\theta + 2\lambda)}$$

$$D_{11} = 4\lambda^2 (\theta + 2\lambda), \; D_{12} = \frac{R_1}{S_1}, \; D_{21} = \frac{R_3}{S_2}, \; D_{22} = \frac{R_5}{S_4}$$

$$R_1 = \theta(\theta + 2\lambda) + 4\lambda^2, \; R_2 = (\theta + 2\lambda)(\beta + 4\lambda) + 2\lambda\beta, \; R_3 = S_2 - 2\lambda\theta\beta, \; R_4 = S_2 + 4\lambda^2\theta, \; R_5 = (\beta + 2\lambda)^2.\theta^2 + \beta S_2$$

$$S_1 = 2\lambda\theta(\theta + 2\lambda), \; S_2 = 2\lambda(\theta + 2\lambda)(\beta + 2\lambda), \; S_3 = 2\lambda S_2, \; S_4 = \theta\beta S_2$$

CONCLUSION

Figure 3 shows that the difference (T_1-T_2) of mean time to system failure decreases rapidly as the failure rate (λ) increases for fixed values of other parameters. The values of T_1 and T_2 become more or less equal when $\lambda > 0.15$ otherwise $T_1 > T_2$. Also this difference becomes too high with the increase of repair rate θ. Figure 4 reflects the variation of availability difference (A_1-A_2) and it is observed that (A_1-A_2) keeps on increasing as the failure rate (λ) increases but it decreases with the increase of repair rate θ. Behaviour of the profit difference (P_1-P_2) is shown in figure 5. It is seen that $P_1 < P_2$ for $\lambda < 0.095$ when $\theta = 0.8$ otherwise $P_1 > P_2$. The difference (P_1-P_2) has the maximum negative value when $0.1 < \lambda < 0.035$ and this becomes less as repair rate θ increases. Further, mode II still remains profitable over mode I for $\lambda < 0.098$ with the increase of repair rate θ from 0.8 to 1.2 while if $\lambda > 0.098$, $P_1 > P_2$ for $\theta = 1.2$. Hence finally we conclude that model I is more reliable to use and thus profitable in normal conditions. However, model II can be made profitable over model I by taking unit of low failure rate and server of high repair rate.

ILLUSTRATION

Let us consider the real life data of an Ash handling plant of a fertilizer Industry
Failure rate of ash handling water pump = 0.000234 per unit time
Repair rate = 0.8 per unit time, Waiting time = 50 units.
The application of above, suggested system models, approach yields following results;

For model 1:
MTSF = 2927184.409 unit time, Availability = 0.99762, Busy period = 0.0005850,
Expected Number of Visits per unit time = 0.0004677, Profit = 4987.77258 per unit time

For model 2:
MTSF = 3599287.388 unit time, Availability = 0.99996, Busy period = 0.0012046,
Expected Number of Visits per unit time = 0.0009631, Profit = 4999.27561 per unit time

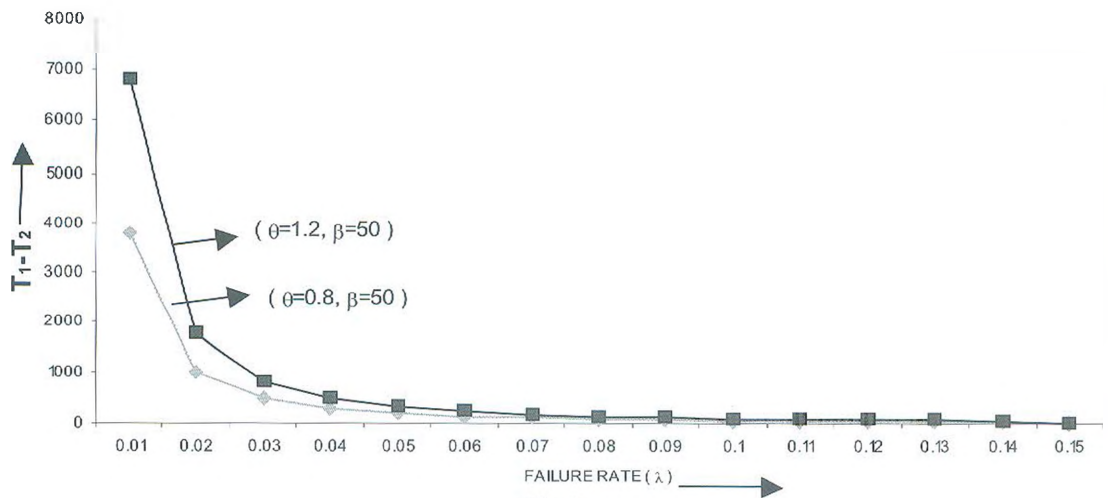**GRAPH : MTSF DIFFERENCE v/s FAILURE RATE**

Fig.3

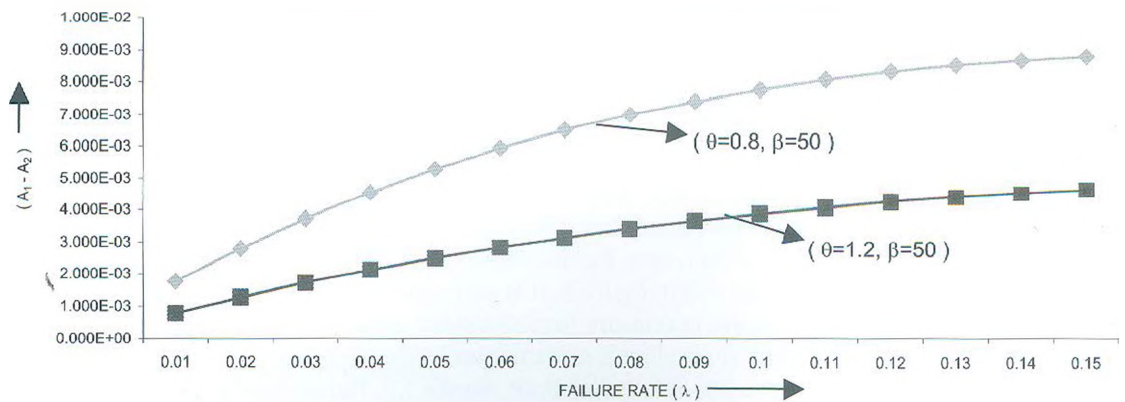**GRAPH: AVAILABILITY DIFFERENCE V/S FAILURE RATE**

Fig. 4

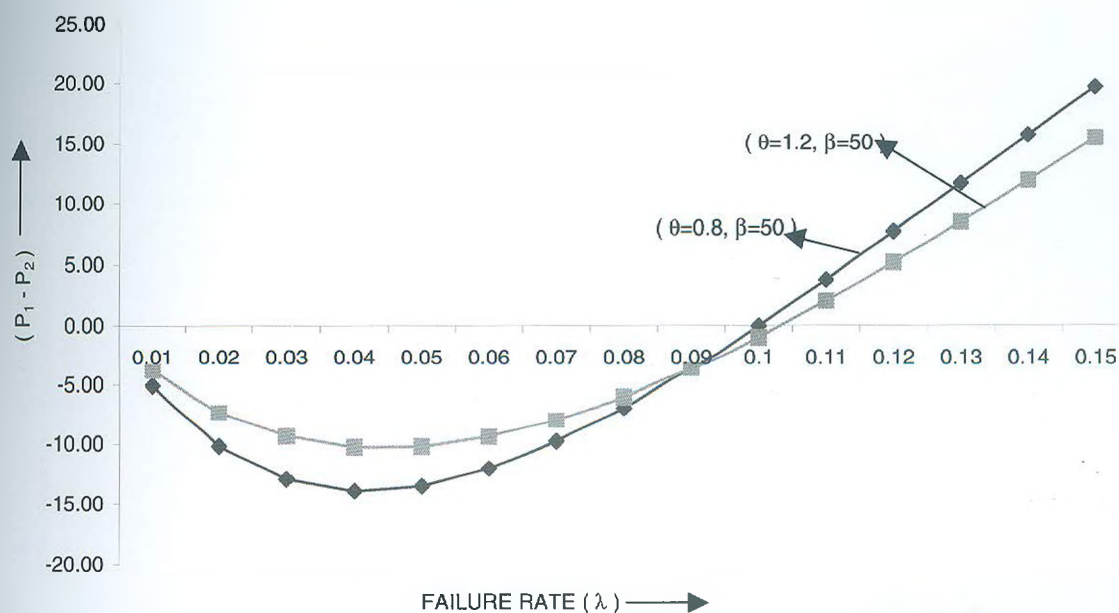GRAPH : PROFIT DIFFERENCE v/s FAILURE RATE

Fig. 5

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